Maximal Abelian Torsion Subgroups of Diff(C,0)

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Abstract. *In the study of the local dynamics of a germ of diffeomorphism fixing the origin in* $\mathbf{C}$*, an important problem is to determine the centralizer of the germ in the group Diff(*$\mathbf{C}$*,0) of germs of diffeomorphisms fixing the origin. When the germ is not of finite order, then the centralizer is abelian, and hence a maximal abelian subgroup of Diff(*$\mathbf{C}$*,0). Conversely any maximal abelian subgroup which contains an element of infinite order is equal to the centralizer of that element. A natural question is whether every maximal abelian subgroup contains an element of infinite order, or whether there exist maximal abelian torsion subgroups; we show that such subgroups do indeed exist, and moreover that any infinite subgroup of the rationals modulo the integers* $\mathbf{Q}/\mathbf{Z}$ *can be embedded into Diff(*$\mathbf{C}$*,0) as such a subgroup.*

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1. Introduction.

We consider the group of germs of diffeomorphisms in $\mathbf{C}$ fixing the origin, Diff($\mathbf{C}$,0) = $\{f : f(z) = \lambda z + O(z^2), \lambda \neq 0\}$. The local dynamics of such germs f near the fixed point 0 has been intensively studied, in particular the question, when is f *linearizable*, i.e. conjugate to its linear part $L_\lambda(z) = \lambda z$. If the fixed point is *attracting* ($|\lambda| < 1$) or *repelling* ($|\lambda| > 1$) then a classical theorem of Koenigs asserts that f is linearizable. When the fixed point is *indifferent* ($|\lambda| = 1, \lambda = e^{2\pi i\alpha}, \alpha \in (\mathbf{R}/\mathbf{Z})$, the linearizability of f depends very sensitively on the arithmetic of the *rotation number* α. Any *nondegenerate parabolic* germ ($\alpha = p/q \in \mathbf{Q}/\mathbf{Z}$, $f^q \neq id$) is nonlinearizable whereas a *degenerate parabolic* germ ($\alpha = p/q \in \mathbf{Q}/\mathbf{Z}$, $f^q = id$) is always linearizable. When α is irrational, for α poorly approximable by rationals all germs f are linearizable whereas for α very well approximable by rationals there exist nonlinearizable germs. The sharp arithmetic condition is called the Brjuno condition ([Br]); its optimality was shown by Yoccoz ([Yo]) (see for example [PM1] for a survey of the linearization problem).

The centralizer Cent(f) = $\{g : g \circ f = f \circ g\}$ of a germ f in Diff($\mathbf{C}$,0) can be thought of as the group of symmetries of the dynamics (it's elements conjugate the dynamics to itself). The centralizer clearly contains any abelian subgroup containing f; when f is not of finite order, it is well known that Cent(f) is abelian (we recall the description of such centralizers in the following section), and is hence a maximal abelian subgroup of Diff($\mathbf{C}$,0).

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Moreover when $f'(0) = \lambda$ is not a root of unity then the group homomorphism given by the "rotation number map"

$$\rho : \mathrm{Diff}(\mathbf{C}, 0) \to \mathbf{C}/\mathbf{Z}$$
$$g \mapsto \frac{1}{2\pi i} \log g'(0)$$

is injective restricted to Cent(f), identifying the centralizer with a subgroup of $\mathbf{C}/\mathbf{Z}$. The restriction to Cent(f) is surjective if and only if f is linearizable, whereas if f is nonlinearizable, then $\rho(\mathrm{Cent}(f)) \subset \mathbf{R}/\mathbf{Z}$. Understanding the arithmetic of subgroups of $\mathbf{R}/\mathbf{Z}$ which occur as groups of rotation numbers $\rho(\mathrm{Cent}(f))$ in the nonlinearizable case can thus be seen as a generalization of the linearization problem. This seems to be a very difficult problem for which few results are known: Moser [Mo] has shown that the irrationals occurring in such subgroups must admit good simultaneous rational approximations, while Perez-Marco [PM2] has constructed examples where the subgroups are uncountable, containing Cantor sets.

Any maximal abelian subgroup of Diff($\mathbf{C}$,0) containing an infinite order element f is equal to Cent(f). In order to classify all maximal abelian subgroups of Diff($\mathbf{C}$,0), it is natural to ask therefore whether there exist maximal abelian torsion subgroups (otherwise only centralizers can occur). We show that this is indeed the case, and moreover any infinite subgroup of $\mathbf{Q}/\mathbf{Z}$ can occur as the corresponding group of rotation numbers:

Theorem 1. *For any infinite subgroup H of $\mathbf{Q}/\mathbf{Z}$, there exists a maximal abelian torsion subgroup $\hat{H}$ of Diff($\mathbf{C}$,0) such that ρ restricted to $\hat{H}$ is injective, and $\rho(\hat{H}) = H$.*

Thus the rotation numbers of maximal abelian torsion subgroups can be arbitrary and are not subject to any arithmetic condition.

2. Centralizers in Diff(C,0).

We recall some well known facts about centralizers of elements of Diff($\mathbf{C}$,0). We denote by $\mathbf{C}[[z]]$ the ring of formal power series in z and by $\mathrm{Diff}_{For}(\mathbf{C},0) := \{ f \in \mathbf{C}[[z]] : f(0) = 0, f'(0) \neq 0 \}$ the group of formal diffeomorphisms of germs fixing 0. We identify Diff($\mathbf{C}$,0) is with the subgroup of elements of $\mathrm{Diff}_{For}(\mathbf{C},0)$ whose series converge. For $f \in$ Diff($\mathbf{C}$,0) we denote its centralizer in $\mathrm{Diff}_{For}(\mathbf{C},0)$ by $\mathrm{Cent}_{For}(f)$; the analytic centralizer Cent(f) is then identified with the subgroup of elements of the formal centralizer $\mathrm{Cent}_{For}(f)$ whose series converge.

Proposition 2. *For any $\lambda \in \mathbf{C}^*$ which is not a root of unity, $\mathrm{Cent}(L_\lambda) = \mathrm{Cent}_{For}(L_\lambda) = \{ L_\mu \}_{\mu \in \mathbf{C}^*}$.*

Proof: Comparing coefficients of both sides of $g \circ L_\lambda = L_\lambda \circ g$ gives $\lambda^n g_n = \lambda g_n, n \geq 1$ (where $g(z) = \sum_n g_n z^n$), so $g_n = 0$ for $n \geq 2$. $\diamond$

We recall that when λ is not a root of unity, any $f(z) = \lambda z + f_2 z^2 + \ldots \in \mathrm{Diff}_{For}(\mathbf{C},0)$ is formally linearizable; there exists a unique formal germ $h_f(z) = z + h_2 z^2 + \ldots \in \mathrm{Diff}_{For}(\mathbf{C},0)$

conjugating f to L_λ, and f is linearizable if and only if the formal series h_f is convergent. Indeed comparing coefficients of both sides of the conjugacy equation $h_f \circ f = L_\lambda \circ h_f$ determines a recursive solution of the form $h_n = \frac{P_n(\lambda, f_2, \ldots, f_n, h_1, \ldots h_{n-1})}{(\lambda^n - \lambda)}, n \geq 2$, where the P_n's are universal polynomials.

Observing that any germ (formal or analytic) conjugating two germs also conjugates their centralizers, it follows from the previous proposition that

Proposition 3. *For any $f(z) = \lambda z + O(z^2) \in$ Diff($\mathbf{C}$,0) with λ not a root of unity, $Cent_{For}(f) = \{h_f \circ L_\mu \circ h_f^{-1} : \mu \in \mathbf{C}^*\}$. If*
(i) f is linearizable then Cent(f) $= Cent_{For}(f) = \{h_f \circ L_\mu \circ h_f^{-1} : \mu \in \mathbf{C}^*\}$ and $\rho :$Cent(f) $\to \mathbf{C}/\mathbf{Z}$ is an isomorphism.
(ii) f is nonlinearizable then Cent(f) $= \{h_f \circ L_\mu \circ h_f^{-1} : \mu \in \mathbf{C}^*$ such that $h_f \circ L_\mu \circ h_f^{-1}$ converges$\}$, all germs in Cent(f) are indifferent and $\rho :$Cent(f) $\to \mathbf{R}/\mathbf{Z}$ is an injective homomorphism.

In either case above we see that the centralizer is abelian (and hence a maximal abelian subgroup of Diff($\mathbf{C}$,0)). In the case of a nondegenerate parabolic germ f, while f is not (even formally) linearizable, it can always be formally embedded into the flow of a holomorphic vector field with a zero at the origin. Indeed for $n \geq 1, \tau \in \mathbf{C}$, the vector field $X_{n,\tau} = \frac{z^{n+1}}{1+\tau z^n}\frac{\partial}{\partial z}$ is a normal form for any holomorphic vector field with a zero of order (n+1) at the origin, and if $\rho(f) = p/q, f^q(z) = z + az^{n+1} + \ldots$ with $a \neq 0$, then for some $\tau = \tau(f)$ there exists a formal germ ϕ such that $f = \phi \circ e^{2\pi i p/q}\exp(X_{n,\tau}) \circ \phi^{-1}$. The formal and analytic centralizers of $\exp(X_{n,\tau})$ are equal, both given by the abelian group of germs $\{ e^{2\pi i k/n}\exp(tX_{n,\tau}) : k \in \mathbf{Z}/n\mathbf{Z}, t \in \mathbf{C} \} \cong \mathbf{Z}/n\mathbf{Z} \times \mathbf{C}$ (note the rotations around the origin of finite order n commute with the flow of $X_{n,\tau}$). For these classical results we refer the reader to the articles of Baker ([Ba]), Ecalle ([Ec]) and Voronin ([Vo]). They allow us to describe the centralizer of nondegenerate parabolic germs as follows:

Proposition 4. *For any nondegenerate parabolic germ f such that $\rho(f) = p/q, f^q(z) = z + az^{n+1} + \ldots, a \neq 0$, for some $\tau \in \mathbf{C}$, Cent(f) is given by the subgroup of elements of $\{ \phi \circ e^{2\pi i k/n}\exp(tX_{n,\tau}) \circ \phi^{-1} : k \in \mathbf{Z}/n\mathbf{Z}, t \in \mathbf{C} \}$ which converge (where $\phi \in$Diff$_{For}$($\mathbf{C}$,0) formally conjugates f to $e^{2\pi i p/q}\exp(X_{n,\tau})$).*

Thus the centralizer is again abelian (isomorphic to a subgroup of $\mathbf{Z}/n\mathbf{Z} \times \mathbf{C}$), hence maximal abelian, all elements of the centralizer are parabolic, and $\rho(Cent(f)) \subseteq \left(\frac{1}{n}\mathbf{Z}\right)/\mathbf{Z}$ is finite.

3. Maximal Abelian Torsion Subgroups of Diff(C,0).

For $r > 0$ we denote by $\mathbf{D}_r$ the disc of radius r centered around the origin. For any simply connected domain D containing 0 and $\alpha \in \mathbf{R}/\mathbf{Z}$, we denote by $R_{D,\alpha}$ the intrinsic rotation of D around 0 by angle $2\pi\alpha$, i.e. the unique automorphism $R : D \to D$ such that $R(0) = 0, R'(0) = e^{2\pi i \alpha}$, which can be described as the conjugate of the rigid rotation $R_\alpha(z) = e^{2\pi i \alpha}z$ by any Riemann mapping $h : \mathbf{D} \to D$ such that $h(0) = 0$. The intrinsic

rotations of a given domain form a group isomorphic to $\mathbf{R}/\mathbf{Z}$, and any conformal mapping between two simply connected domains which fixes the origin conjugates their intrinsic rotations. The construction of maximal abelian torsion subgroups rests on the following

Proposition 5. *Let $r, M, \delta > 0$ be real and $q, a \geq 2$ integers. Given $f \in$Diff($\mathbf{C}$,0) such that $\rho(f) = 1/q, f^q = id$, there exists $\phi \in$Diff($\mathbf{C}$,0), $\phi = \phi(r, M, \delta, a, f)$ such that $\rho(\phi) = 1/(aq), f = \phi^a$, and for any $g \in$Diff($\mathbf{C}$,0), if g, g^{-1} are univalent on $\mathbf{D}_r$, $|g'(0)|, |(g^{-1})'(0)| \leq M$, and g commutes with ϕ, then $\rho(g) \in \left(\frac{1}{q}\mathbf{Z} + \mathbf{D}_\delta\right)/\mathbf{Z}$ (where $\frac{1}{q}\mathbf{Z}+\mathbf{D}_\delta$ denotes numbers of the form $k/q + \tau, k \in \mathbf{Z}$, $|\tau| < \delta$).*

Proof: The idea is roughly, given f, to find an ϕ satisfying $f = \phi^a$ and having singularities very close to the origin which are almost symmetrically distributed with respect to the rotation $R_{\frac{1}{q}}$, so that any germ commuting with ϕ must preserve the singularities and hence have rotation number close to a multiple of $1/q$. We achieve this as follows:

Fix a linearization $h(z) = z + O(z^2)$ of f, so $f = h^{-1} \circ R_{1/q} \circ h$. Let $\epsilon_0, \epsilon_1 > 0$ be such that h^{-1} is univalent on $\mathbf{D}_{\epsilon_0}$ and $\mathbf{D}_{\epsilon_1} \subset U = h^{-1}(\mathbf{D}_{\epsilon_0}) \subset \mathbf{D}_r$. For $0 < \epsilon < \epsilon_0$ let $D(\epsilon, q)$ be the slit domain $D(\epsilon, q) := \mathbf{D}_{\epsilon_0} - \cup_{j \in \mathbf{Z}/q\mathbf{Z}} \{t\, e^{2\pi i j/q} : \epsilon \leq t < \epsilon_0\}$. Note $R_{1/q}(D(\epsilon, q)) = D(\epsilon, q)$, so $R_{D(\epsilon,q),1/q} = R_{1/q} = h \circ f \circ h^{-1}$; thus if we set $\phi := h^{-1} \circ R_{D(\epsilon,q),1/(aq)} \circ h$ then $f = \phi^a$. We check that for ϵ small enough, depending only on r, δ, a and h, the germ ϕ satisfies the conclusion asserted by the Proposition:

The domain $V_\epsilon := h^{-1}(D(\epsilon, q))$ is a slit domain equal to U minus a finite union of slits $(\gamma_j)_{j \in \mathbf{Z}/q\mathbf{Z}}$, which is invariant under ϕ. Let $z_j = h^{-1}(\epsilon\, e^{2\pi i j/q})$ be the endpoints of the slits γ_j. Then any $z^* \in \gamma_j$ distinct from z_j is biaccessible from V_ϵ, and there are two paths β, β' in V_ϵ landing at z^* such that any conformal representation from V_ϵ to $\mathbf{D}$ tends to distinct points of $\partial \mathbf{D}$ as z tends to z^* along β, β'. In the plane of $w = h(z)$, the corresponding point $w^* = h(z^*)$ is biaccessible from $D(\epsilon, q)$, and any intrinsic rotation $R_{D(\epsilon,q),\alpha}$ with $\alpha \notin \frac{1}{q}\mathbf{Z}/\mathbf{Z}$ tends to distinct limits in $\partial D(\epsilon, q)$ as $w = h(z)$ tends to $w^* = h(z^*)$ along $h(\beta), h(\beta')$. So the conjugate germ ϕ will tend to distinct limits contained in $\partial V_\epsilon \subset \mathbf{D}_r$ as z tends to z^* along β, β'.

Now let $g(z) = \mu z + O(z^2)$ be a germ such that g, g^{-1} are univalent in $\mathbf{D}_r$ and $|g'(0)|, |(g^{-1})'(0)| \leq M$. By classical results on univalent functions, the family $\mathcal{F}_{M,r}$ of such functions is a normal family, so for ϵ small enough depending only on r, M, $g(\mathbf{D}_{2\epsilon}) \subset \mathbf{D}_{\epsilon_1} \subset U$. Suppose g commutes with ϕ. Then taking $z^* \in \gamma_j \cap \mathbf{D}_{2\epsilon}, z^* \neq z_j$, and letting z tend to z^* along β, β' in the equation $\phi(g(z)) = g(\phi(z))$, since the RHS tends to two distinct limits, we must have $g(z^*) \in \gamma_{j'}$ for some j' (otherwise the LHS would tend to a unique limit). It follows that $g(\mathbf{D}_{2\epsilon} \cap (\cup_j \gamma_j)) \subseteq g(\mathbf{D}_{2\epsilon}) \cap (\cup_j \gamma_j)$, applying the same argument to g^{-1} gives the reverse inclusion, so $g(\mathbf{D}_{2\epsilon} \cap (\cup_j \gamma_j)) = g(\mathbf{D}_{2\epsilon}) \cap (\cup_j \gamma_j)$. In particular, for any j, $g(z_j) = z_{j'}$ for some j'. So

$$\frac{g(z_j)}{z_j} = \frac{z_{j'}}{z_j} = \frac{h(\epsilon\, e^{2\pi i j'/q})}{h(\epsilon\, e^{2\pi i j/q})}$$

Since g belongs to the normal family $\mathcal{F}_{M,r}$, we have a uniform estimate $|g(z_j)/z_j - \mu| \leq C|z_j|$ where the constant C only depends on r, M and not on g. It is clear that $|z_j| = O(\epsilon)$

and that the RHS is equal to $e^{2\pi i(j'-j)/q} + O(\epsilon)$ with the constants in the $O(\epsilon)$ terms only depending on h; it follows that

$$\mu = e^{2\pi i(j'-j)/q} + O(\epsilon)$$

with the constant in the error term $O(\epsilon)$ only depending on r, M and h, so for ϵ small enough, depending only on these parameters and δ but not on g, we will have

$$\rho(g) = \frac{1}{2\pi i}\log\mu \in \left(\frac{(j'-j)}{q} + \mathbf{D}_\delta\right)/\mathbf{Z} \qquad \diamond .$$

We need the following simple description of subgroups of $\mathbf{Q}/\mathbf{Z}$:

Proposition 6. *(i) Any finite nonzero subgroup H of $\mathbf{Q}/\mathbf{Z}$ is cyclic, $H =< 1/q >$ for some $q \geq 2$.*
(ii) Any infinite subgroup H of $\mathbf{Q}/\mathbf{Z}$ is an increasing union of cyclic subgroups, $H = \cup_n < 1/q_n >$ for some sequence (q_n) such that $q_n | q_{n+1}$ for all n.

Proof: (i) Let $x_0 = p/q \in H$, $(p, q$ coprime, $q \geq 2)$ be such that $d(x_0, \mathbf{Z}) = \mathrm{Min}_{x \in H, x \neq 0} d(x, \mathbf{Z})$. Then since $< p/q >=< 1/q >$ in $\mathbf{Q}/\mathbf{Z}$, we must have $x_0 = \pm 1/q$. For any $x = p'/q' \in H(p', q'$ coprime, $q' \geq 2)$, $< 1/q' >=< p'/q' >\subseteq H$ and we can write $1/q' = a/q + r$ for some integer a and some $0 \leq r < 1/q$. Then $r = 1/q' - a/q \in H$ and $d(r, \mathbf{Z}) = r < 1/q = d(x_0, \mathbf{Z})$ so $r = 0$, hence $1/q' = a/q$ and $x \in< x_0 >$. Thus $H =< 1/q >$.

(ii) Let $(x_n)_{n \geq 1}$ be an enumeration of H. Then H is the increasing union of the finite subgroups $H_n =< x_1, \ldots, x_n >$, each of which is of the form $H_n =< 1/q_n >$ by (i), and $q_n = O(H_n)$ divides $q_{n+1} = O(H_{n+1})$ since H_n is a subgroup of H_{n+1}. $\diamond$

We can now construct maximal abelian torsion subgroups as follows:

Proof of Theorem 1: Given an infinite subgroup H of $\mathbf{Q}/\mathbf{Z}$, write it in the form $H = \cup_{n \geq 1} < 1/q_n >$ where $q_{n+1} = a_{n+1} q_n$ for some integers $a_n \geq 2$ (we may assume the cyclic groups are strictly increasing).

Fix a monotone decreasing sequence (r_n) converging to 0, and an increasing sequence M_n converging to $+\infty$. Let $\delta_n = \frac{1}{3}\frac{1}{q_{n+1}}$. It is easy to check that for this choice of δ_n,

$$\bigcap_{n \geq N} \left(\frac{1}{q_n}\mathbf{Z} + \mathbf{D}_{\delta_n}\right)/\mathbf{Z} = \left(\frac{1}{q_N}\mathbf{Z}\right)/\mathbf{Z}$$

for any $N \geq 1$.

Let $f_1 = R_{1/q_1}$ and for $n \geq 1$ define f_{n+1} inductively by $f_{n+1} = \phi(r_n, M_n, \delta_n, a_{n+1}, f_n)$ (where ϕ is the germ given by Proposition 5). Then $\rho(f_n) = 1/q_n, f_n = f_{n+1}^{a_{n+1}}$, so the increasing union of cyclic subgroups $\hat{H} := \cup_{n \geq 1} < f_n >$ is an abelian torsion subgroup of Diff($\mathbf{C}$,0) such that $\rho(\hat{H}) = H$. We check that $\hat{H}$ is maximal abelian:

Let g be a germ commuting with f_n for all n. Let $N = N(g) \geq 1$ be such that g and g^{-1} are univalent on $\mathbf{D}_{r_n}$ and $|g'(0)|, |(g^{-1})'(0)| \leq M_n$ for all $n \geq N$. By the choice of the f_n's and Proposition 5, it follows that

$$\rho(g) \in \bigcap_{n \geq N} \left(\frac{1}{q_n}\mathbf{Z} + \mathbf{D}_{\delta_n} \right) /\mathbf{Z} = \left(\frac{1}{q_N}\mathbf{Z} \right) /\mathbf{Z}$$

Thus g is parabolic. Since $\rho(\mathrm{Cent}(g)) \supseteq \rho(\hat{H}) = H$ is infinite, g must be a degenerate parabolic, $g^{q_N} = id$. If $\rho(g) = k/q_N$, then the germ $g \circ f_{q_N}^{-k}$ is tangent to the identity, of finite order (since g, f_{q_N} commute and are of finite order), and hence must be the identity. So $g = f_{q_N}^k \in \hat{H}$. $\diamond$

To summarize, we can classify maximal abelian subgroups of Diff($\mathbf{C}$,0) according to their groups of rotation numbers as follows:

Theorem 7. *Any maximal abelian subgroup $\hat{H}$ of Diff($\mathbf{C}$,0) must be of one of the following kinds (and all kinds occur): either*
(i) $\rho(\hat{H})$ is a finite subgroup of $\mathbf{Q}/\mathbf{Z}$ in which case $\hat{H}$ contains a nondegenerate parabolic germ f, is equal to Cent(f), and is isomorphic to a subgroup of $\mathbf{Z}/n\mathbf{Z} \times \mathbf{C}$,
or
(ii) $\rho(\hat{H})$ is an infinite subgroup of $\mathbf{Q}/\mathbf{Z}$ in which case all elements of $\hat{H}$ are degenerate parabolic germs, and $\hat{H}$ is isomorphic to the subgroup $\rho(\hat{H})$ of $\mathbf{Q}/\mathbf{Z}$,
or
(iii) $\rho(\hat{H}) \cap (\mathbf{R} - \mathbf{Q})/\mathbf{Z} \neq \emptyset$, in which case $\hat{H}$ contains an irrationally indifferent germ f, is equal to Cent(f) and isomorphic to either $\mathbf{C}/\mathbf{Z}$ or to a subgroup of $\mathbf{R}/\mathbf{Z}$.

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